Announcements

1) Math Colloquium

3-4, CB 2064

"p-adic numbers"

Arbitrary Metric Space Definitions for Sequences of Functions

Let (X, d_1) and (Y, d_2) be metric spaces. A sequence of functions $(f_n)_{n=1}^{\infty}$ where $f_n : X \to Y$ is said to converge pointwise to $f : X \to Y$ if $\forall\, x \in X,\ \varepsilon > 0\ \exists\, N \in \mathbb{N}$

$$d_2(f_n(x), f(x)) < \varepsilon \quad \forall\, n \geq N.$$

$(f_n)_{n=1}^{\infty}$ converges to f

uniformly if $\forall\ \varepsilon > 0$

$\exists\ N \in \mathbb{N}$ such that

$\forall\ x \in X$,

$d_2(f_n(x), f(x)) < \varepsilon$

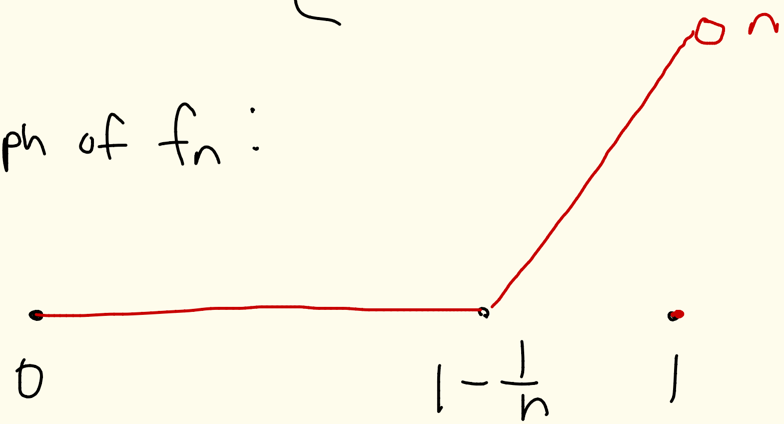
$\forall\ n \geq N$.

Example 1:

$$f_n : [0,1] \to \mathbb{R}$$

$$f_n(x) = \begin{cases} 0, & 0 \le x \le 1 - \frac{1}{n} \text{ or } x = 1 \\ n^2 x + (n - n^2), & 1 - \frac{1}{n} < x < 1 \end{cases}$$

graph of f_n:



Exactly the same argument from last class gives that

$f_n \to 0$ pointwise.

But $\displaystyle\int_0^1 f_n(x)\,dx = \frac{1}{2}$

$\forall\, n \in \mathbb{N}$ and

$\displaystyle\int_0^1 0\, dx = 0.$

This shows that if convergence is only pointwise, it need not be the case that

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx \quad \text{equals}$$

$$\int_a^b \lim_{n\to\infty} f_n(x)\,dx$$

Observe that the sequential example from the previous class has

$$\int_0^1 f_n(x)\,dx = \frac{1}{2n} \to 0$$

as $n \to \infty$, so uniform convergence is a sufficient but not necessary condition for the integrals to converge to the integral of the limit.

Theorem: (continuity)

Let (X, d_1) and (Y, d_2) be metric spaces. Then if $f, f_n : X \to Y$ $\forall n \in \mathbb{N}$, $f_n \to f$ uniformly and f_n is continuous at $x \in S$, then f is continuous at $x \in S$.

Proof: Suppose f_n is continuous at $x \in X$ and $f_n \to f$ uniformly.

We want, given $\varepsilon > 0$, to show that $\exists\ \delta > 0$ such that $\forall\ y \in B(x, \delta)$,

$$d_2(f(x), f(y)) < \varepsilon.$$

But

$$d_2(f(x), f(y))$$

$$\leq d_2(f(x), f_n(x)) + d_2(f_n(x), f(y))$$

$$\leq d_2(f(x), f_n(x)) + d_2(f_n(x), f_n(y)) + d_2(f_n(y), f(y)).$$

Then $\exists\ N \in \mathbb{N}$ so that $\forall$ $z \in X$, $d_2(f_n(z), f(z)) < \frac{\varepsilon}{3}$ $\forall$ $n \geq N$.

Since f_n is continuous at x $\forall\, n \in \mathbb{N}$, then for each $n \geq N$, $\exists$ $\delta_n > 0$ such that if $d_1(x,y) < \delta_n$ then $d_2(f_n(x), f_n(y)) < \frac{\varepsilon}{3}$.

Pick an $n \geq N$ and let $\delta = \delta_n$.

Then $\forall\ y \in X$ with $d_1(x,y) < \delta$, we have

$$d_2(f(x), f(y))$$

$$\leq d_2(f_n(x), f(x)) + d_2(f_n(x), f_n(y))$$

$$+ d_2(f_n(y), f(y))$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

and so f is continuous at x. $\square$

Lemma: (Maxes and mins)

Suppose $S \subseteq \mathbb{R}$ and

$f_n \to f$ uniformly on S.

Suppose f, f_n are bounded

$\forall\, n \in \mathbb{N}$. Then if

$$M = \sup_{x \in S} f(x)$$

$$m = \inf_{x \in S} f(x)$$

$$M_n = \sup_{x \in S} f_n(x)$$

$$m_n = \inf_{x \in S} f_n(x),$$ then

$m_n \to m$ and $M_n \to M$.

Proof: Let $M = \sup_{x \in S} f(x)$

and let $\varepsilon > 0$. Then

$\exists\ y \in S$, $\quad M - \frac{\varepsilon}{2} < f(y)$.

Then

$$M < f(y) + \frac{\varepsilon}{2}$$

$$= f(y) - f_n(y) + f_n(y) + \frac{\varepsilon}{2}$$

Since $f_n \to f$ uniformly, $\exists$

$N \in \mathbb{N}$ such that

$$|f(x) - f_n(x)| < \varepsilon \quad \forall n \geq N$$

We have

$$M \leq f(y) - f_n(y) + f_n(y) + \frac{\varepsilon}{2}$$

$$\leq f(y) - f_n(y) + M_n + \frac{\varepsilon}{2}$$

$$\Rightarrow$$

$$M - M_n < f(y) - f_n(y) + \frac{\varepsilon}{2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\forall \; n \geq N.$$

Now for each $n \in \mathbb{N}$

$\exists \; y_n \in S$,

$$M_n - \frac{\varepsilon}{2} < f_n(y_n)$$

Then

$$\begin{aligned} M_n &< f_n(y_n) + \frac{\varepsilon}{2} \\ &= f_n(y_n) - f(y_n) + f(y_n) + \frac{\varepsilon}{2} \\ &\leq f_n(y_n) - f(y_n) + M + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + M + \frac{\varepsilon}{2} = M + \varepsilon \end{aligned}$$

if $n \geq N$.

This implies

$M_n - M < \varepsilon$. Combining

the two inequalities,

we have

$$|M - M_n| < \varepsilon$$

$\Rightarrow$ $M_n \to M$. A similar

proof shows $m_n \to m$. $\square$

Theorem: (Integration)

Suppose $f, f_n : [a,b] \to \mathbb{R}$ $\forall\ n \in \mathbb{N}$ and that f_n is integrable $\forall\ n \in \mathbb{N}$.

Then if $f_n \to f$ uniformly on $[a,b]$, we have that f is integrable and that

$$\boxed{\lim_{n\to\infty} \int_a^b f_n(x) = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx}$$

Proof: Let $\varepsilon > 0$.

Let P be any partition of $[a,b]$. Then

$$U(f,P) - L(f,P)$$

$$= \sum_{i=1}^{k-1} (M_i - m_i)(x_{i+1} - x_i)$$

$$= \sum_{i=1}^{k-1} \left(\left(M_i^n - m_i^n\right) + \left(M_i - M_i^n\right) + \left(m_i^n - m_i\right) \right) \cdot (x_{i+1} - x_i))$$

where

$$M_i = \sup_{x \in [x_i, x_{i+1}]} f(x)$$

$$m_i = \inf_{x \in [x_i, x_{i+1}]} f(x)$$

$$M_i^n = \sup_{x \in [x_i, x_{i+1}]} f_n(x)$$

$$m_i^n = \inf_{x \in [x_i, x_{i+1}]} f_n(x).$$

Since $f_n \rightrightarrows f$ uniformly,

we may choose $N \in \mathbb{N}$

so that $|f_n(x) - f(x)| < \frac{\varepsilon}{6(b-a)}$

for all $n \geq N$.

Fix an $n \geq N$, choose

a partition P with

$$U(f_n, P) - L(f_n, P) < \frac{\varepsilon}{3}.$$

So then

$$U(f,P)-L(f,P)$$

$$= \sum_{i=1}^{k} (M_i^n - m_i^n)(x_{i+1}-x_i) < \frac{\varepsilon}{3}$$

$$+ \sum_{i=1}^{k} \underbrace{(m_i^n - m_i)}_{< \frac{\varepsilon}{3(b-a)}}(x_{i+1}-x_i)$$

$$+ \sum_{i=1}^{k} \underbrace{(M_i - M_i^n)}_{< \varepsilon/3(b-a)}(x_{i+1}-x_i)$$

By the proof of the previous lemma, $M_i - M_i^n,\ m_i^n - m_i < \frac{\varepsilon}{3(b-a)}$

$\forall\ 1 \leq i \leq K.$

Then

$$U(f,P) - L(f,P)$$

$$< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3(b-a)} \sum_{i=1}^{k} (x_{i+1} - x_i)$$

$$= \frac{\varepsilon}{3} + \frac{2}{3} \frac{\varepsilon}{b-a} \cdot (b-a)$$

$$= \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon \text{, which}$$

shows f is integrable.

The proof also shows

$$\lim_{n \to \infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx. \quad \square$$